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Quasi-multiplications and inertial automorphisms (I)

by

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The purpose of this note is to introduce a definition of "inertial automorphism" on an arbitrary commutative ring with unity which reduces to the old definition in the case R is a complete discrete valuation ring. We generalize the notion of a value non-decreasing mapping on a valuation ring to the concept of "quasi-multiplication on a ring". We observe that rings R are embeddable in the ring of "quasi-multiplications" on R. Using this notion we develop some induced homomorphism theorems (theorems 2 & 3). We define inertial automorphisms as automorphisms which are also quasi-multiplications. We generalize the notion of "valuation-ring" to M-ring, where M is a chain of ideals with valuation-like properties. The strong 3rd condition in the definition was needed to give the result of theorem 4. Finally we begin a study of certain classes of subrings defined by the M-structure and the automorphism structure which have proven important in the case of valuation rings.

Rings of quasi-multiplications

Suppose R is a commutative ring with identity, then a ring R_Q containing R is a ring of quasi-multiplications on R if any ideal of R is an ideal of R_Q as well.

If R is a commutative ring with identity and R_Q a ring of quasi-multiplications on R, then $y \in R_Q$ and $x \in R$, implies $y(x) \subset (x)$, i.e., $yx = ux$, $u \in R$. If we let $f^*: R \to R$ be defined by $f^*(x) = u$, then it follows that $yx = xf^*(x)$, i.e., we can regard y as a multiplication of x "by a function on R", hence the name quasi-multiplication.

Notice that if we define a function $f: R \to R$ to be a quasi-multiplication if there is a function $f^*: R \to R$ such that $f(x) = xf^*(x)$, then $y \in R_Q$ implies "multiplication by y" is a quasi-multiplication. In the situation where R is a valuation-ring with valuation V then a function $f: R \to R$ is a quasi-multiplication if and only if f is value non-decreasing, i.e., $V(f(x)) \geq V(x)$ for all x. In this sense we can view quasi-multiplications as natural generalizations of value non-decreasing functions on a valuation ring to arbitrary commutative rings with identity.

<u>Lemma 1</u>: Suppose R is an arbitrary commutative ring with identity, then the collection $R_{[Q]}$ of all quasi-multiplications is a ring under the regular definitions of operator addition and multiplication. $R_{[Q]}$ has identity I, $I(x) = x$.

<u>Proof</u>: Suppose $f, g \in R_{[Q]}$, $x \in R$, then

$(f + g)(x) = f(x) + g(x) = xf^*(x) + xg^*(x) = x(f^* + g^*)(x)$

and $f + g \in R_{[Q]}$.

Furthermore, $(fg)(x) = f(g(x)) = g(x)f^*(g(x)) = xg^*(x)f^*(g(x))$

and $fg \in R_{[Q]}$.

<u>Lemma 2</u>: If on $R_{[Q]}$ we define $(f*g)(x) = f(x)g(x)$, then $R_{[Q]}$ becomes a commutative ring.

Proof:

$(f * g)(x) = f(x)g(x) = xf^*(x)g(x) = xf(x)g^*(x)$ and $f * g \in R_{[Q]}$.
Since R is commutative it follows that $f * g = g * f$.

We'll denote the ring in lemma 1 by $R_{<Q>}$ and the ring in lemma 2 by $R_{<<Q>>}$.

Lemma 3: If $y \in R$, let $M_y: R \to R$ be defined by $M_y(x) = y(x)$. Then the mapping $\phi: R \to R_{<Q>}$ defined by $\phi(y) = M_y$ is an isomorphism.

Proof:

That $\phi(y_1 + y_2) = \phi(y_1) + \phi(y_2)$ is obvious.
Next, observe that
$\phi(y_1 y_2) = M_{y_1 y_2} = M_{y_1} M_{y_2} = \phi(y_1)\phi(y_2)$.

Also $\phi(y) = 0$ implies $yx = 0$ for all x. Since R has an identity we obtain that $y1 = y = 0$ and ϕ is an isomorphism.

Theorem 1: Suppose R_Q is a ring of quasi-multiplications on R, then R_Q can be "embedded" in $R_{<Q>}$.

Proof:

Let $y \in R_Q$, then letting $f_y: R \to R$ be defined by $yx = f_y(x)$ we get a mapping $\phi: R_Q \to R_{<Q>}$.
That ϕ is a homomorphism is clear.
Suppose $\phi(y) = 0$, then $yx = 0$ for all $x \in R$. Thus Ker ϕ = Annihilator of R in R_Q. It is clear that $R_Q/\text{Ker } \phi$ is a ring of quasi-multiplications on R (R contains 1, hence $M_y \notin \text{Ker } \phi$ for $y \neq 0$!) and on $R_Q/\text{Ker } \phi$ the mapping constructed above is an isomorphism.

From now on we will always assume that a ring R_Q of quasi-multiplications on R has annihilator (0) so that theorem 1 will hold universally, i.e., any ring R_Q of quasi-multiplications will be regarded as a subring of $R_{<Q>}$ via the natural isomorphism constructed in Theorem 1. Notice that since $I = m_1$ any ring R_Q of quasi-multiplications will also be a ring with identity. Notice that $R_{<Q>}$ according to the definitions really is a ring of quasi-multiplications on R. Notice further that $R_{<Q>}$ is a two-sided R-module, i.e., its structure as a left R-module coincides with its structure as a right R-module. This follows from the fact that R is a commutative ring. Thus define $(rf)(x) = rf(x) = f(x)r = (fr)(x)$. Notice that <u>as a ring operation</u> $(fr)(x) = (fm_r)(x) = f(rx) \neq f(x)r$ in general! To avoid confusion we shall always use $R_{<Q>}$ as a left R-module.

<u>Theorem 2</u>: If R_1, R_2 are commutative rings with identity and $\nu: R_1 \to R_2$ is a homomorphism, then $\nu^*: R_{1<Q>} \to R_{2<Q>}$ define defined by

$(\nu^*(f))(\nu(y)) = \nu(f(y))$ is a homomorphism into.

<u>Proof</u>:

Ker ν is an ideal of R_1 thus for any element $f \in R_{1<Q>}$ it is true that $f(\text{Ker } \nu) \subset \text{Ker } \nu$. Thus if $y \in \text{Ker } \nu$, then $(\nu^*(f))(\nu(y)) = (\nu^*(f))(0) = \nu(f(y)) = 0$.
Furthermore, $(\nu^*(f_1 + f_2))(\nu(y)) = \nu((f_1 + f_2)(y)) =$
$= \nu(f_1(y) + f_2(y)) = \nu(f_1(y)) + \nu(f_2(y)) = (\nu^*(f_1))(\nu(y)) +$
$+ (\nu^*(f_2))(\nu(y))$.
Similarly, $(\nu^*(f_1f_2))(\nu(y)) = \nu((f_1f_2)(y)) = \nu(f_1(f_2(y))) =$
$= (\nu^*(f_1))(\nu(f_2(y))) = \nu^*(f_1)(\nu^*(f_2)(\nu(y))) =$
$= \nu^*(f_1)\nu^*(f_2)(\nu(y))$. Hence the theorem follows.

<u>Theorem 3</u>: If $\nu: R_1 \to R_2$ has the property that $\text{Ker } \nu \subset \bigcap_{x \notin \text{Ker } \nu} (x)$, then ν^* is onto.

<u>Proof</u>:

Indeed, let $\bar{f}: R_2 \to R_2$ be a quasi-multiplication.
Define $f: R_1 \to R_1$ as follows. Let $f(\text{Ker } \nu) = 0$ and if $x \notin \text{Ker } \nu$, select $f(x) \in \nu^{-1}(\bar{f}(\nu(x)))$ arbitrarily.
We claim that $f: R_1 \to R_1$ is a quasi-multiplication.

Indeed since $\overline{f}$ is a quasi-multiplication we have $\overline{f}(\nu(x)) = \nu(x)\ \overline{f}^*(\nu(x))$. Thus if $y \in \nu^{-1}(\overline{f}(x))$, we get $\nu(y) = \nu(x)\ \overline{f}^*(\nu(x)) = \nu(x)\nu(z) = \nu(xz)$.
Thus $y \in (x) + \text{Ker } \nu = (x)$ since Ker $\nu \subset (x)$.
Hence $f(x) = xf^*(x)$ for $x \notin \text{Ker } \nu$, $f(x) = x \cdot 0$ for $x \in \text{Ker } \nu$.
Thus is f indeed a quasi-multiplication. By construction we get $(\nu^*(f))(\nu(y)) = \nu(f(y)) = \overline{f}(\nu(y))$, i.e., $\nu^*(f) = \overline{f}$ and ν^* is onto.

Corollary: If R_1 is a valuation ring then ν^* is onto.

We are now ready to define the concept of inertial isomorphism on an arbitrary commutative ring with identity. Suppose R is such a ring, then an inertial isomorphism $\sigma: R \to R$ is an isomorphism which is a quasi-multiplication on R.
Notice that if R is a complete valuation ring, then an isomorphism is an inertial isomorphism if and only if it is value preserving, i.e., value non-decreasing, i.e., a quasi-multiplication on R.
The inertial automorphisms serve as a group of units in $R_{\langle Q\rangle}$, a subgroup of the group of units of $R_{\langle Q\rangle}$.
We shall denote the group of inertial isomorphisms on R by G_I.

In the next section we will discuss a type of ring in which we have the following situation:

(1) A chain of ideals $\{\overline{m}_n\}_{n=1}^{\infty}$ with $\overline{m}_{i+1} \subset \overline{m}_i$.
(2) $\bigcap_{i\in\omega} \overline{m}_i = (0)$.

We'll call this ring an M-ring if in addition the following condition is satisfied

(3) For every $x \neq 0$ $\exists$ an $N(x) < \infty$ such that $\overline{m}_{N(x)} \subset (x)$.

Notice that if R is a valuation ring with value group Z, i.e., a discrete valuation ring and if $V(\pi) = 1$, then letting $\overline{m}_n = (\pi)^n = (\pi^n)$, $M = \{\overline{m}_n\}_{n=1}^{\infty}$, we get that R is an M-ring.

Suppose now that R is an M-ring $M = \{\overline{m}_n\}_{n=1}^{\infty}$, then the M-<u>pseudo-ramification groups</u> G_n are defined as follows:

$$G_n = \{\sigma \in G_I \mid \sigma(x) - x \in \overline{m}_n\} .$$

Again notice that if R is a complete discrete valuation ring, then if $M = \{\overline{m}_n = (\pi^n), V(\pi) = 1\}$, the M-pseudo-ramification groups G_n are just the ordinary pseudo-ramification group.

<u>M-rings and completions</u>

Suppose R is a commutative ring with identity which is an M-ring with respect to a collection of ideals $M = \{\overline{m}_n\}_{n=1}^{\infty}$.

<u>Definition 1</u>: A sequence of functions $\{f_\mu\}_{\mu=1}^{\infty}$ is a null-sequence if given $N > 0 \ \exists \ \mu(N) \ni \mu \geq \mu(N) \Rightarrow: f_\mu: R \to \overline{m}_N$.

Notice that any null-sequence is "eventually" a quasi-multiplication i.e., given x there is a μ such that $f_\mu(x) \in (x)$.
Indeed, suppose we take $N(x)$ as in condition (3) and pick $\mu \geq \mu(N(x))$, then $f_\mu: R \to (x)$ and $f_\mu(x) \in (x)$.

Next we say:

<u>Definition 2</u>: A sequence of functions $\{f_\mu\}_{\mu=1}^{\infty}$ is a limiting sequence if there is a function f such that $\{f'_\mu = f_\mu - f\}_{\mu=1}^{\infty}$ is a null-sequence.

Proposition 1: If $\{f'_\mu = f_\mu - f'\}_{\mu=1}^\infty$ and $\{f''_\mu = f_\mu - f''\}_{\mu=1}^\infty$ are null-sequences, then $f' = f''$.

Proof:

Pick $\mu \geq \mu(N)$, then $f_\mu - f': R \to \overline{m}_N$ and $f_\mu - f'': R \to \overline{m}_N$ (actually $\mu(N) = \max(\mu_1(N), \mu_2(N))$).
Hence $f' - f'': R \to \overline{m}_N$. Since this is independent of μ, we get $f' - f'': R \to \bigcap_{n\in\omega} \overline{m}_n = (0)$ and $f' = f''$.

Thus limiting sequences have unique limits indicated with $\lim_\mu f_\mu$.

Definition 3: A sequence of functions $\{f_\mu\}_{\mu=1}^\infty$ is Cauchy if given N there is a $\mu(N)$ such that $\mu_1, \mu_2 > \mu(N)$ implies $f_{\mu_1} - f_{\mu_2}: R \to \overline{m}_N$.

Proposition 2: If a sequence is limiting, then it is Cauchy.

Proof:

If $\{f_\mu\}_{\mu=1}^\infty$ is limiting, suppose $\lim_\mu f_\mu = f$ and $\mu > \mu(N) \Rightarrow f_\mu - f: R \to \overline{m}_N$. Then $\mu_1, \mu_2 > \mu(N) \Rightarrow f_{\mu_1} - f_{\mu_2} = (f_{\mu_1} - f) + (f - f_{\mu_2}): R \to \overline{m}_N$ and $\{f_\mu\}_{\mu=1}^\infty$ is Cauchy.

The converse is true only under special assumptions on R.

Definition 4: A sequence $\{x_\mu\}_{\mu=1}^\infty$ of elements is limiting in case the sequence of functions $\{f_\mu : f_\mu(x) = x_\mu\}_{\mu=1}^\infty$ is limiting.

Definition 5: An M-ring is complete if every Cauchy sequence of constant functions $\{f_\mu : f_\mu(x) = x_\mu\}_{\mu=1}^\infty$ is limiting.

Proposition 3: If R is a complete M-ring, then a Cauchy sequence is necessarily limiting.
If R is not complete, then not every Cauchy sequence is limiting.

Proof:

Suppose R is a complete M-ring, then $\{f_\mu\}_{\mu=1}^{\infty}$ a Cauchy-sequence implies $\{f_\mu(x)\}_{\mu=1}^{\infty}$ a Cauchy-sequence of elements, hence necessarily limiting. Let $f(x) = \lim_\mu f_\mu(x)$.
Then $\{f_\mu - f\}_{\mu=1}^{\infty}$ is a null-sequence and hence $\lim_\mu f_\mu = f$.
If R is not complete, then suppose $\{x_\mu\}_{\mu=1}^{\infty}$ is a Cauchy-sequence which is not limiting. Then $\{f_\mu : f_\mu(x) = x_\mu\}_{\mu=1}^{\infty}$ is a Cauchy-sequence of functions which is not limiting.

We note that if R is an M-ring and R_Q is a ring of quasi-multiplications on R, then R_Q is an M-ring for the same family of ideals $M = \{\overline{m}_n\}_{n=1}^{\infty}$ of R regarded as ideals of R_Q.

Theorem 4: If R is a complete M-ring then $R_{<Q>}$ is also a complete M-ring.

Proof:

Suppose $\{f_\mu\}_{\mu=1}^{\infty}$ is a Cauchy-sequence of quasi-multiplications. Since R is a complete M-ring $\{f_\mu\}_{\mu=1}^{\infty}$ is limiting, let $f = \lim_\mu f_\mu$.
Let $N(x)$ be such that $\overline{m}_{N(x)} \subset (x)$, then $\mu \geq \mu(N(x))$
$\Rightarrow (f - f_\mu)(x) \in \overline{m}_{N(x)} \subset (x)$. Thus $f(x) = f_\mu(x) + x\rho_\mu(x) =$
$= xf_\mu^*(x) + x\rho_\mu(x) = x(f_\mu^*(x) + \rho_\mu(x))$ and f is a quasi-multiplication. Thus it follows that $R_{<Q>}$ is a complete M-ring.

The Inertial Subring of a Ring

Let R be a commutative ring with identity, then let $R_0 = \{x \mid \sigma(x) = x \text{ for all } \sigma \in G_I\}$. Then we obtain R_0 as a subring of R. The inertial subring of R.

If $\overline{m}$ is an ideal of R, then we can construct:

$\nu_m : R \to R/\bar{m}$, let $R_{0;\bar{m}} = R_0/\bar{m} \subset R/\bar{m}$.

Proposition 4: If R is an M-ring and R is complete then R_0 is complete.

Proof:

Let $\{x_\mu\}_{\mu=1}^{\infty}$ be a Cauchy-sequence in R_0. Let $x = \lim_\mu x_\mu$, $x \in R$.
Then we have $\{\sigma(x_\mu)\}_{\mu=1}^{\infty} = \{x_\mu\}_{\mu=1}^{\infty}$ for all $\sigma \in G_I$.
Hence $x = \lim_\mu \sigma(x_\mu)$.

But, $\sigma(x - x_\mu) = \sigma(x) - \sigma(x_\mu) \in \bar{m}_N$ if $\mu > \mu(N)$.
Thus $\sigma(x) - x = \sigma(x) - \sigma(x_\mu) + \sigma(x_\mu) - x_\mu \in \bar{m}_N$ if $\mu > \mu(N)$.
Since thus $\sigma(x) - x \in \bigcap_{n \in \omega} \bar{m}_n = (0)$ we have $\sigma(x) = x$, i.e., $x \in R_0$ and R_0 is complete.

If R is an M-ring say $x \in R$ has index of inertia relative to M equal to N if $\sigma(x) - x \in \bar{m}_N$ for all $\sigma \in G_I$ but there is a σ^* such that $\sigma(x) - x \notin \bar{m}_{N+1}$. Denote this index by $\Delta_M(x)$.

Lemma 4: $R_0 = \{x \mid \Delta_M(x) = \infty\}$.

Proof:

If $x \in R_0$, then $\sigma(x) - x \in \bar{m}_N$ for all N and $\Delta_M(x) = \infty$.
If $\Delta_M(x) = \infty$, then $\sigma(x) - x \in \bigcap_{n \in \omega} \bar{m}_n = (0)$ and $x \in R_0$.

Note that the index-of-inertia on R_0 is independent of the system M which makes R an M-ring.

Proposition 5: Suppose R is an intergral domain and x is integral over R_0, if $\sigma \in G_I$ then ${}^{\sigma(x)}/x$ is a root of unity in R. If x has degree n over R_0, then ${}^{\sigma(x)}/x$ is an n^{th} root of unity in R.

Proof:

Let K be the qoutient field of R and K^* an algebraic closure of K. Suppose $x^n + a_1 x^{n-1} + \ldots + a_n = 0$, $a_n \neq 0$, $a_i \in R_0$.

We have $\prod_{i=1}^{n} (x - w_i) = 0$, where w_i, $i = 1, \ldots, n$ are the roots.

If $\sigma \in G_I$ extend σ to σ^* on K^*, then

$$\prod_{i=1}^{n} (\sigma(x) - \sigma^*(w_i)) = 0 .$$

Since $x = x \cdot x_\sigma$, x_σ a unit in R.

We get: $x_\sigma^n \prod_{i=1}^{n} (x - \frac{\sigma^*(w_i)}{x_\sigma}) = 0$ or $\prod_{i=1}^{n} (x - \frac{\sigma^*(w_i)}{x_\sigma}) = 0$.

Since $a_n \neq 0$, $a_n = \prod_{i=1}^{n} w_j = \prod_{i=1}^{n} \sigma^*(w_j) = x_\sigma^n \prod_{i=1}^{n} \frac{\sigma^*(w_j)}{x_\sigma} =$

$= x_\sigma^n \prod_{i=1}^{n} w_{j(i)} = x_\sigma^n a_n$, we obtain $x_\sigma^n = 1$.

Thus $x_\sigma = \frac{\sigma(x)}{x}$ is an $n^{\underline{th}}$ root of unity.

<u>Corollary 1</u>: If x has degree n over R_0, then $x^n \in R_0$.

<u>Proof</u>:

Suppose x has degree n, then $(\frac{\sigma(x)}{x})^n = 1$.

Hence, $\frac{\sigma(x)^n}{x^n} = \frac{\sigma(x^n)}{x^n} = 1$, i.e., $\sigma(x^n) = x^n$ for all $\sigma \in G_I$.

Thus $x^n \in R_0$.

<u>Corollary 2</u>: If R is a characteristic 0 integral domain and $D : R \to R$ a derivation, then $D(R_0) = 0 \Rightarrow D(\overline{R}_0) = 0$ where $\overline{R}_0$ is the integral closure of R_0 in R.

<u>Proof</u>:

$x \in \overline{R}_0$ implies $x^n \in R_0$ for some n. Thus $D(x^n) = nx^{n-1}D(x) = 0$ and $nx^{n-1} \neq 0$ implies $D(x) = 0$.

Suppose $\overline{m} \subset R_0$ is an ideal, then we define

$$\sqrt[R]{\overline{m}} = \{x \in R \mid x^n \in \overline{m}\}.$$

(i) $\sqrt[R]{\bar{m}} = \sqrt[\bar{R}_0]{\bar{m}}$

(ii) $\sqrt[\bar{R}_0]{\bar{m}}$ is an ideal in $\bar{R}_0$

(iii) $\sqrt[\bar{R}_0]{\sqrt[\bar{R}_0]{\bar{m}}} = \sqrt[\bar{R}_0]{\bar{m}}$, i.e., $\sqrt[\bar{R}_0]{\bar{m}}$ is a radical ideal

$$x^n \in \sqrt[\bar{R}_0]{\sqrt[\bar{R}_0]{\bar{m}}} \Rightarrow (x^n)^p \in \bar{m} \Rightarrow x \in \sqrt[\bar{R}_0]{\bar{m}}$$

(iv) If $\bar{m}$ is prime, then $\sqrt[\bar{R}_0]{\bar{m}}$ is prime

Suppose $x \notin \sqrt[\bar{R}_0]{\bar{m}}$, then $x^n \in R_0 \Rightarrow x^n \notin \bar{m}$.

Suppose $xy \in \sqrt[\bar{R}_0]{\bar{m}}$, then $(xy)^s \in \bar{m} \Rightarrow (xy)^{sn} = (x^{sn})(y^{sn}) \in \bar{m}$.

If $y^t \in R_0$, then $(x^{snt})(y^{snt}) \in \bar{m}$, x^{snt}, $y^{snt} \in R_0$, $x^{snt} \notin \bar{m}$.

Thus $y^{snt} \in \bar{m}$ and $y \in \sqrt[\bar{R}_0]{\bar{m}}$, i.e., $\sqrt[\bar{R}_0]{\bar{m}}$ is prime.

(v) If $\bar{m}$ is $\bar{p}$-primary, then $\sqrt[\bar{R}_0]{\bar{m}}$ is prime.

If $\bar{m}$ is $\bar{p}$-primary, then $\sqrt{\bar{m}} = \bar{p}$ and $\sqrt[\bar{R}_0]{\sqrt{\bar{m}}} = \sqrt[\bar{R}_0]{\bar{p}}$ is prime.

But $\sqrt[\bar{R}_0]{\sqrt{\bar{m}}} = \sqrt[\bar{R}_0]{\bar{m}}$.

(vi) If $\bar{m}$ is an ideal, then $\sqrt[\bar{R}_0]{\bar{m}} \cap R_0 = \sqrt{\bar{m}}$.

<u>Proposition 6</u>: If $\sigma \in G_I$, then $\sigma/\bar{R}_0$ is an inertial automorphism on $\bar{R}_0$.

<u>Proof</u>:

Let $x \in \bar{R}_0$, then $\sigma(x) = x \cdot x_\sigma$ with x_σ an n^{th} root of unity. Since $1 \in R_0$, we have $x_\sigma \in \bar{R}_0$ and $\sigma/\bar{R}_0$ is an inertial automorphism on R_0.

Pseudo-inertial subrings of M-rings

Now suppose R is an M-ring, $M = \{\overline{m}_n\}_{n=1}^{\infty}$.
We define rings

$$R_n = \{x \mid \Delta_M(x) \geq n\}.$$

Notice that R_n is indeed a ring. If $x, y \in R_n$, $\sigma \in G_I$, then
$\sigma(x + y) - (x + y) = (\sigma(x) - x) + (\sigma(y) - y) \in \overline{m}_n$,
$\sigma(xy) - xy = \sigma(x)\sigma(y) - xy = \sigma(x)\sigma(y) - \sigma(x)y + \sigma(x)y - xy =$
$= \sigma(x)(\sigma(y) - y) + (\sigma(x) - x)y \in \overline{m}_n$.

Lemma 5: $R_n \supset \overline{m}_n$; $R_n \supset R_{n+1}$.
Proof:

$\sigma \in G_I \Rightarrow \sigma(\overline{m}_n) \subset \overline{m}_n$ and $x \in \overline{m}_n \Rightarrow \sigma(x) - x \in \overline{m}_n$.
$R_n \supset R_{n+1}$ obviously.

Lemma 6: $\bigcap_{n \in \omega} R_n = R_0$.
Proof:

Lemma 12.

Lemma 7: If R is a complete M-ring, then R_n is complete.
Proof:

Suppose that $\{x_\mu\}_{\mu=1}^{\infty} \subset R_n$ is a Cauchy sequence.
Let $\sigma \in G_I$. Then $\mu, \nu > \mu(N) \Rightarrow x_\mu - x_\nu \in \overline{m}_N$,
thus $\sigma(x_\mu - x_\nu) = \sigma(x_\mu) - \sigma(x_\nu) \in \overline{m}_N$, since $\sigma \in G_I$.
Hence $\{\sigma(x_\mu)\}_{\mu=1}^{\infty}$ is Cauchy. We have $\sigma(x) = \lim_\mu \sigma(x_\mu)$.
Select μ_0 such that $\sigma(x) - \sigma(x_\mu) \in \overline{m}_n$ for all $\mu \geq \mu_0$.
Then $\sigma(x_\mu) \in \overline{m}_n$ ($x_\mu \in R_n$!) $\Rightarrow \sigma(x) \in \overline{m}_n$ and $x \in R_n$.
Thus the result follows.

We shall call the rings R_n pseudo-inertial subrings of R_n.

Suppose R is an M-ring, $M = \{\overline{m}_n\}_{n=1}^{\infty}$. Let G_n be the pseudo-inertial groups.

Let $R_{[n]} = \{x \mid \sigma(x) = x \ \forall \sigma \in G_n\}$.
Then $x, y \in R_{[n]}$, $\sigma(x + y) = \sigma(x) + \sigma(y) = x + y$,
$\sigma(xy) = \sigma(x)\sigma(y) = xy$. Thus is $R_{[n]}$ a ring.

Lemma 8: $R_{[n]} \subset R_{[n+1]}$.
Proof:

$G_{n+1} \triangleleft G_n$.

Lemma 9: If R is a complete M-ring, then $R_{[n]}$ is complete.
Proof:

Suppose that $\{x_\mu\}_{\mu=1}^{\infty} \subset R_{[n]}$ is a Cauchy-sequence.
If $x = \lim_\mu x_\mu$ and $\sigma \in G_n$, then $\sigma(x) = \lim_\mu \sigma(x_\mu) = \lim_\mu x_\mu = x$ and $x \in R_{[n]}$. Thus the lemma follows.

Lemma 10: Suppose R is an integral domain and x is integral over $R_{[m]}$, if $\sigma \in G_m$, then ${}^{\sigma(x)}/x$ is a root of unity in R.
If x has degree n over $R_{[m]}$, then ${}^{\sigma(x)}/x$ is an $n^{\underline{th}}$ root of unity in R.

Proof:

The proof is exactly the same as the proof of proposition 5.

Corollary 1: If x has degree n over $R_{[m]}$, then $x^n \in R_{[m]}$.

Corollary 2: If R is a characteristic 0 integral domain and $D : R \to R$ a derivation, then $D(R_{[m]}) = 0 \Rightarrow: D(\overline{R}_{[m]}) = 0$, where $\overline{R}_{[m]}$ is the integral closure of $R_{[m]}$ in R.

Corollary 3: Remark (i) - (vi) of the previous section hold for ideals in $R_{[m]}$.

Corollary 4: If $\sigma \in G_m$, then $\sigma \mid \overline{R}_{[m]}$ is an inertial automorphism.

We shall call the rings $R_{[m]}$ pseudo-inertial subrings of the $2^{\underline{nd}}$ kind.

References

[1] N. Heerema, Derivations on p-adic fields, TAMS 102 (1962), pp. 346-351;

[2] S. MacLane, Subfields and automorphism groups of p-adic fields, Ann. of Math., 40 (1939), pp. 423-442;

[3] J. Neggers, Derivations on $\bar{p}$-adic fields (to appear);

[4] O.F.G. Schilling, The theory of Valuations, Mathematical Surveys, No. 4, New York, 1950;

[5] O. Zariski and P. Samuel, Commutative Algebra, Vols. 1 and 2, D. Van Nostrand Co., Princeton, New Jersey.